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On the central limit theorem for weakly dependent sequences with a decomposed strong mixing coefficient

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Weak invariance principles are established for strictly stationary weakly dependent sequences, having a decomposed strong mixing coefficient into two parts, one based on the strong mixing condition with a polynomial mixing rate and other based on the ρ-mixing condition. The result is applied to the output of the Tukey '3R smoother'.

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strictly stationary sequence * strong mixing conditions * weak invariance principle

1. Introduction

Suppose $X=\{X_k\}_{k\in\mathbb{Z}}$ is a strictly stationary sequence of random variables on a probability space (Ω, K, P). Define $\mathcal{P}_0=\mathcal{P}_0(X)=\sigma(X_k: k\leqslant 0)$ the past of the process until the moment of time 0 and $\mathcal{F}_n=\mathcal{F}_n(X):=\sigma(X_k: k\geqslant n)$ the future after n steps. For each $n\geqslant 1$ define

$$\alpha(n):=\sup|P(A\cap B)-P(A)P(B)|,\quad A\in\mathcal{P}_0,\ B\in\mathcal{F}_n,$$

$$\rho(n):=\sup|\operatorname{corr}(f,g)|,\quad f\in\mathcal{L}_2(\mathcal{P}_0),\ g\in\mathcal{L}_2(\mathcal{F}_n).$$

The sequence is said to be 'strongly mixing' if $\alpha(n)\to 0$ as $n\to\infty$, 'ρ-mixing' if $\rho(n)\to 0$ as $n\to\infty$. It is easy to see that ρ-mixing implies strong mixing.

The ρ-mixing condition can be reformulated in terms of pair of events (see Bradley, 1983; or Bradley and Bryc, 1985).

For each $n\geqslant 1$ define

$$\rho_0(n)=\sup\frac{|P(A\cap B)-P(A)P(B)|}{[P(A)P(B)]^{1/2}},\quad A\in\mathcal{P}_0,\ B\in\mathcal{F}_n,\ P(A)P(B)>0.$$

Then $\rho_0(n)\to 0$ as $n\to\infty$ is equivalent to $\rho(n)\to 0$ as $n\to\infty$.

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Mixing types of dependence lead to many useful limit theorems with broad applicability in statistical mechanics (see, e.g., Denker and Philipp, 1984) or statistics in general.

Let us denote by $S_n = X_1 + X_2 + \cdots + X_n$, $\sigma_n^2 = \mathrm{Var}(S_n)$ and $W_n(t) = S_{[nt]}/\sigma_n$, where $[x]$ denotes the integer part of x. We also denote by W the standard Brownian motion on $[0, 1]$ and the weak convergence by $\Rightarrow$. In order to establish the central limit theorem for strongly mixing sequences, the most instrumental way is to find a good estimate for the rate of convergence of $\alpha(n)$ to zero as we can see in the following theorem due to Ibragimov (1962) (see Ibragimov and Linnik, 1971, Theorem 18.5.3) and to Oodaira and Yoshihara (1972).

Theorem A. *Suppose $\{X_n\}$ is strongly mixing strictly stationary, centered sequence of random variables, such that for some $\delta > 0$, $E|X_1|^{2+\delta} < \infty$ and $\sum_{n=1}^{\infty} \alpha(n)^{\delta/(2+\delta)} < \infty$. Then, $\sigma^2 := \lim_{n\to\infty} \sigma_n^2/n$ exists, $0 \leq \sigma^2 < \infty$. If in addition $\lim_{n\to\infty} \sigma_n^2/n > 0$, then $W_n \Rightarrow W$.* □

In some situations it is difficult to compute the speed of convergence to 0 imposed to $\alpha(n)$ and one has to look at an alternative approach such as verifying the ρ-mixing condition, perhaps without concern for mixing rate (see Peligrad, 1986, for a survey on C.L.T. under mixing conditions). When this approach fails there is another possibility: for every n fixed to construct two large events D'_n and D''_n with $P(D'_n \cap D''_n) \to 1$ as $n \to \infty$, and such that, $\rho_n(\mathcal{P}_0|D'_n, \mathcal{F}_n|D''_n) \to 0$ as $n \to \infty$. In this way, the strong mixing coefficient appears decomposed into two parts, a pure strong mixing part, related to the probability of the complement of $D'_n \cap D''_n$ and a ρ-mixing part. To be more specific this situation implies:

Definition 1.1. We say that he strictly stationary sequence X satisfies a decomposed strong mixing condition with the coefficients $\bar{\gamma}(n) = (\tilde{\alpha}(n), \tilde{\lambda}(n))$ if $\tilde{\alpha}(n) \geq 0, \tilde{\lambda}(n) \geq 0$ for every $n \geq 1$, $\lim_{n\to\infty} \bar{\gamma}(n) = 0$ and

$$|P(A \cap B) - P(A)P(B)| \leq \tilde{\alpha}(n) + \tilde{\lambda}(n)[P(A)P(B)]^{1/2} \tag{1.1}$$

for every $A \in \mathcal{P}_0$ and $B \in \mathcal{F}_n$ and for every $n \geq 1$.

The decomposition of the strong mixing coefficient is useful for the stiuation when we can compute only the speed of convergence to 0 of $\tilde{\alpha}(n)$ while $\tilde{\lambda}(n)$ tends to 0 as $n \to \infty$ arbitrarily slowly. While the condition $\bar{\gamma}(n) \to 0$ is too weak for many useful results we shall prove that some additional information only on the pure strong mixing part $\tilde{\alpha}(n)$ can decide in C.L.T. and its weak invariance principle. This theorem can be applied in many situations because in some examples a decomposition of the strong mixing coefficients is obtained at no extra cost in the process of estimating the strong mixing coefficients.

In Section 3 of this paper we discuss a statistical smoothing procedure based on the Tukey '3R smoother' which provides us with a class of examples of sequences satisfying a decomposed strong mixing condition. Actually our paper was motivated by this example.

The notion of decomposed mixing conditions was introduced in Bradley and Peligrad (1986), where a weak invariance principle was obtained under a polynomial rate imposed only to $\tilde{\alpha}(n)$. But that condition on $\tilde{\alpha}(n)$ is too restrictive and cannot be applied to the above mentioned example.

We shall prove here a new theorem for mixing sequences satisfying (1.1), which can be applied to the example from Section 3 and at the same time gives a new result for strongly mixing sequences.

The technique used is a new one, based on maximal inequalities.

Theorem 1.1. *Assume that* $\{X_n\}$ *is a strictly stationary sequence of random variables with* $EX_1=0$ *and for some* $\delta>0$, $E|X_1|^{2+\delta}<\infty$. *Assume also that* X *satisfies* (1.1) *with* $\tilde{\lambda}(n)\to 0$ *and, for some* $\beta>(2+\delta)/\delta$, $\tilde{\alpha}(n)=O(n^{-\beta})$ *as* $n\to\infty$.

Assume in addition that σ_n^2 *has the representation* $\sigma_n^2=nh(n)$ *where* $h(x)$ *is a function which is slowly varying when* $x\to\infty$. *Then* $W_n\Rightarrow W$.

Notice first that the representation required for σ_n^2 is a necessary condition for the C.L.T. for strongly mixing sequences, when $\sigma_n^2\to\infty$ (see Ibragimov, Linnik, 1971, Theorem 18.1.1). When $\tilde{\alpha}(n)=0$ for every n, this theorem refers to ρ-mixing sequences and contains a theorem in Ibragimov (1975).

For the situation when $\tilde{\lambda}(n)=0$ for every n we find as a corollary the following result:

Corollary 1.2. *Assume* $\{X_n\}$ *is a strictly stationary strongly mixing sequence of random variables such that* $EX_1=0$. *Assume that the following condition holds*: $E|X_1|^{2+\delta}<\infty$ *for some* $\delta>0$ *and for some* $\beta>(2+\delta)/\delta$, $\alpha(n)=O(n^{-\beta})$ *as* $n\to\infty$. *Then,* $W_n\Rightarrow W$ *iff* $\sigma_n^2=nh(n)$, *with* $h(x)$ *a slowly varying function as* $x\to\infty$. □

In this corollary the conditions are as minimal as they can be, as one can see by two results in Bradley (1985). Theorem 3 in Bradley (1985) proves that the condition imposed to $\alpha(n)$ cannot be weakened just to $\sum_{n=1}^{\infty}\alpha(n)^{\delta/(2+\delta)}<\infty$. Moreover, according to Theorem 5 in Bradley (1985) our condition on $\alpha(n)$ does not imply the representation of $\sigma_n^2=nh(n)$ with $h(x)$ as a function slowly varying at infinite.

It is interesting to compare this corollary with Theorem A. Theorem A gives a C.L.T., under the assumption $\lim_{n\to\infty}\sigma_n^2/n\neq 0$. But if $\sigma^2=0$ the additional assumption σ_n^2/n is slowly varying (while approaching 0) as $n\to\infty$ it is not enough to assure C.L.T. in the context of Theorem A, according to Theorem 3 in Bradley (1985). However our corollary shows that if the condition imposed to the mixing coefficients is just marginally stronger than in Theorem A, the C.L.T. continues to hold.

2. Proof of Theorem 1.1 and auxiliary results

In order to prove Theorem 1.1 we shall apply the following theorem from Peligrad (1986), which is a combination of a result of Denker (1986) and Billingsley (1968, Theorem (8.4) and p. 73).

Theorem B. *Let $\{X_n\}$ be a strictly stationary strong mixing, centered sequence of random variables such that $EX_1^2<\infty$ and $\sigma_n^2\to\infty$. In order for $W_n\Rightarrow W$ it is necessary and sufficient that the family $\{S_n^2/\sigma_n^2\}$ is uniformly integrable and for each positive ε there exists $\lambda>1$ such that*

$$P\left(\max_{1\leqslant i\leqslant n}|S_i|>\lambda\sigma_n\right)\leqslant \varepsilon/\lambda^2. \qquad \square$$

By this theorem, we can see that in order to prove Theorem 1.1 it is enough to find a positive real number η, $0<\eta<\delta$ such that $E(\max_{1\leqslant i\leqslant n}|S_i|/\sigma_n)^{2+\eta}$ is a bounded sequence. This will be achieved at the end of several lemmas.

The following lemma is a result from Bradley and Peligrad (1986).

Lemma 2.1. *If X satisfies (1.1) and $1<p\leqslant\infty$, $1<r\leqslant\infty$, $1/p+1/r<1$, $f\in\mathscr{L}_{\max\{p,2\}}(\mathscr{P}_0)$, $g\in\mathscr{L}_{\max\{r,2\}}(\mathscr{F}_n)$. Then*

$$|Efg+FfEg|\leqslant 13(2\tilde\lambda(n))^{1/31}\|f\|_2\|g\|_2+20\tilde\alpha(n)^{1-1/p-1/r}\|f\|_p\|g\|_r. \qquad \square$$

We take now $A\in\mathscr{P}_0$ and $B\in\mathscr{F}_n^\infty$ and put $f=I(A)$, $g=I(B)$ where $I(D)$ denotes the indicator function of a set D. According to Lemma 2.1 we have:

Lemma 2.2. *Assume X satisfies (1.1). Then for every $A\in\mathscr{P}_0$ and $B\in\mathscr{F}_n$*

$$|P(A\cap B)-P(A)P(B)|\leqslant\tilde\rho(n)P^{1/2}(A)P^{1/2}(B)+20\tilde\alpha^{1/q}(n)P^{1/p}(A)$$

for every $1<p\leqslant\infty$, $1<q\leqslant\infty$ such that $1/p+1/q=1$, where $\tilde\rho(n)=13(2\tilde\lambda(n))^{1/31}$. $\square$

We shall establish now the following lemma which is an extension of Ottaviani and Hoffman-Jørgensen's inequalities from the independent case to sequences satisfying (1.1).

Lemma 2.3. *Suppose $\{X_k\}$ is a strictly stationary sequence and suppose that for some r, n, with $n/r\geqslant 2$ and $a>0$, we have*

$$2\max_{2r\leqslant i\leqslant n}P(|S_i|>a)+[2n/r]^{1/2}\tilde\rho(r)\leqslant b<1. \tag{2.1}$$

Suppose p and q are two real numbers greater than 1, such that $1/p+1/q=1$. Then, for every $x\geqslant 5a$ we have

$$P\left(\max_{1\le i\le n}|S_i|>x\right)\le 2(1-b)^{-1}\left[\max_{2r\le i\le n}P(|S_i|>\tfrac{1}{5}x)\right.$$
$$+[n/r]P\left(\max_{1\le i\le 2r}|S_i|>\tfrac{1}{5}x\right)$$
$$\left.+20(n\tilde{\alpha}(r)/r)^{1/q}P^{1/p}\left(\max_{1\le i\le n}|S_i|>x\right)\right] \quad (2.2)$$

and

$$\max_{2r\le i\le n}P(|S_i|>x)\le bP\left(\max_{1\le i\le n}|S_i|>\tfrac{1}{5}x\right)+2[n/r]P\left(\max_{1\le i\le 2r}|S_i|>\tfrac{1}{5}x\right)$$
$$+20(n\tilde{\alpha}(r)/r)^{1/q}P^{1/p}\left(\max_{1\le i\le n}|S_i|>\tfrac{1}{5}x\right). \quad (2.3)$$

Proof. First some notations:

$$M_n=\max_{1\le i\le n}|S_i|, \qquad R_n(x)=P\left(\max_{1\le i\le n}|S_i|>x\right),$$
$$Q_n(x)=P(|S_n|>x), \qquad N_{r,n}(x)=\max_{r\le i\le n}P(|S_i|>x),$$
$$E_i(x)=(M_{i-1}\le x<|S_i|).$$

Let $l=[n/r]$. It is easy to see that

$$P(M_n>x)\le P(|S_n|>\tfrac{1}{5}x)+\sum_{i=0}^{l-2}P\left(\bigcup_{j=1}^{r}E_{ir+j}(x),|S_n-S_{(i+2)r}|>\tfrac{2}{5}x\right)$$
$$+\sum_{i=0}^{l-2}P\left(\bigcup_{j=1}^{r}(E_{ir+j}(x),|S_{(i+2)r}-S_{ir+j}|>\tfrac{2}{5}x)\right)$$
$$+\sum_{i=(l-1)r+1}^{n}P(E_i(x),|S_n-S_i|>\tfrac{4}{5}x).$$

By Lemma 2.2,

$$R_n(x)\le Q_n(\tfrac{1}{5}x)+\sum_{i=0}^{l-2}P\left(\bigcup_{j=1}^{r}E_{ir+j}(x)\right)P(|S_n-S_{(i+2)r}|>\tfrac{2}{5}x)$$
$$+\tilde{\rho}(r)\sum_{i=0}^{l-2}P^{1/2}\left(\bigcup_{j=1}^{r}E_{ir+j}(x)\right)P^{1/2}(|S_n-S_{(i+2)r}|>\tfrac{2}{5}x)$$
$$+20\tilde{\alpha}^{1/q}(r)\sum_{i=0}^{l-2}P^{1/p}\left(\bigcup_{j=1}^{r}E_{ir+j}(x)\right)$$
$$+\sum_{i=0}^{l-2}P\left(\max_{1\le j\le r}|S_{(i+2)r}-S_{ir+j}|>\tfrac{2}{5}x\right)$$
$$+P\left(\max_{(l-1)r+1\le i\le n}|S_n-S_i|>\tfrac{4}{5}x\right).$$

By Cauchy-Schwarz' inequality, stationarity, and a simple computation, we get

$$\begin{aligned}R_n(x) &\leq Q_n(\tfrac{1}{5}x) + 2N_{2r,n}(\tfrac{1}{5}x)R_n(x)\\ &\quad + 2^{1/2}\tilde{\rho}(r)(l-1)^{1/2}R_n^{1/2}(x)N_{2r,n}^{1/2}(\tfrac{1}{5}x)\\ &\quad + 20\tilde{\alpha}^{1/q}(r)(l-1)^{1/q}R_n^{1/p}(x)\\ &\quad + 2(l-1)R_{2r}(\tfrac{1}{5}x) + 2R_{2r}(\tfrac{2}{5}x).\end{aligned}$$

By the obvious inequality $ab \leq a^2 + \frac{1}{4}b^2$ for every real a and b and by (2.1) we get

$$(1-b)R_n(x) \leq \tfrac{5}{4}N_{2r,n}(\tfrac{1}{5}x) + 2lR_{2r}(\tfrac{1}{5}x) + 20\tilde{\alpha}^{1/q}(r)(l-1)^{1/q}R_n^{1/p}(x).$$

This proves (2.2). In order to prove (2.3) let m be an integer such that $2r \leq m \leq n$. Let $p = [m/r]$. We have successively

$$\begin{aligned}Q_m(x) &= P(|S_m| > x,\ M_m > \tfrac{1}{5}x)\\ &\leq \sum_{i=0}^{p-2} P\left(\bigcup_{j=1}^{r} E_{ir+j}(\tfrac{1}{5}x),\ |S_m - S_{(i+2)r}| > \tfrac{2}{5}x\right)\\ &\quad + \sum_{i=0}^{p-2} P\left(\bigcup_{j=1}^{r} (E_{ir+j}(\tfrac{1}{5}x),\ |S_{(i+2)r} - S_{ir+j-1}| > \tfrac{2}{5}x)\right)\\ &\quad + \sum_{i=(p-1)r+1}^{m} P(E_i(\tfrac{1}{5}x),\ |S_m - S_{i-1}| > \tfrac{4}{5}x).\end{aligned}$$

Whence, by Lemma 2.2, Cauchy-Schwarz' inequality, and stationarity, we obtain

$$\begin{aligned}Q_m(x) &\leq 2R_m(\tfrac{1}{5}x)N_{2r,m}(\tfrac{1}{5}x) + [2m/r]^{1/2}\tilde{\rho}(r)R_m^{1/2}(\tfrac{1}{5}x)N_{2r,m}^{1/2}(\tfrac{1}{5}x)\\ &\quad + 20\tilde{\alpha}^{1/q}(r)(p-1)^{1/q}R_m^{1/p}(\tfrac{1}{5}x)\\ &\quad + (p-1)P\left(\max_{0\leq j\leq r}|S_{2r} - S_j| > \tfrac{2}{5}x\right) + 2R_{2r}(\tfrac{1}{5}x)\\ &\leq bR_n(\tfrac{1}{5}x) + 2lR_{2r}(\tfrac{1}{5}x) + 20\tilde{\alpha}^{1/q}(r)(l-1)^{1/q}R_n^{1/p}(\tfrac{1}{5}x).\end{aligned}$$

Now (2.3) follows by taking the maximum for m, $2r \leq m \leq n$. □

By combining (2.2) and (2.3) we get:

Lemma 2.4. *Under the conditions of Lemma* 2.3, *for every* $x \geq 25a$ *we have*

$$\begin{aligned}&P\left(\max_{1\leq i\leq n}|S_i| > x\right)\\ &\quad\leq 2(1-b)^{-1}\left[bP\left(\max_{1\leq i\leq n}|S_i| > \tfrac{1}{25}x\right) + 3[n/r]P\left(\max_{1\leq i\leq 2r}|S_i| > \tfrac{1}{25}x\right)\right.\\ &\qquad\left. + 40(n\tilde{\alpha}(r)/r)^{1/q}P^{1/p}\left(\max_{1\leq i\leq n}|S_i| > \tfrac{1}{25}x\right)\right]. \quad \square \qquad (2.4)\end{aligned}$$

Lemma 2.5. *Suppose $\{X_k\}$ is a strictly stationary sequence such that $E|X_1|^{2+\delta}<\infty$ for some $\delta>0$. Suppose that for some integers n and r with $[n/r]\geq 2$ and $a>0$, (2.1) holds and in addition for a certain η, $0<\eta<\delta$,*

$$d^{-1}:=1-2b(1-b)^{-1}25^{2+\eta}>0. \tag{2.5}$$

(Here b is defined by (2.1).) Then

$$\begin{aligned}EM_n^{2+\eta}&\leq c_1a^{2+\eta}+c_2[n/2r]EM_{2r}^{2+\eta}\\&\quad+c_3a^{2+\eta-(2+\delta)/p}[n\tilde{\alpha}(r)/r]^{1/q}(EM_n^{2+\delta})^{1/p},\end{aligned} \tag{2.6}$$

where p and q are such that $1<p<(2+\delta)/(2+\eta)$, $1/p+1/q=1$. The constants are

$$c_1=d25^{2+\eta},$$
$$c_2=18d(1-b)^{-1}25^{2+\eta},$$
$$c_3=80d(1-b)^{-1}((2+\delta)/p-(2+\eta))^{-1}25^{2+\eta}(2+\eta).$$

Proof. According to Proposition 2.7 of Hoffman-Jørgensen (1974),

$$EM_n^{2+\eta}\leq(25a)^{2+\eta}+(2+\eta)\int_{25a}^{\infty}x^{1+\eta}P(M_n>x)\,\mathrm{d}x.$$

Now, by Lemma 2.4,

$$\begin{aligned}EM_n^{2+\eta}&\leq(25a)^{2+\eta}+2b(1-b)^{-1}25^{2+\eta}EM_n^{2+\eta}\\&\quad+6(1-b)^{-1}25^{2+\eta}[n/r]EM_{2r}^{2+\eta}\\&\quad+80(1-b)^{-1}[n\tilde{\alpha}(r)/r]^{1/q}(2+\eta)\int_{25a}^{\infty}x^{1+\eta}P^{1/p}(M_n>\tfrac{1}{25}x)\,\mathrm{d}x\end{aligned}$$

whence

$$\begin{aligned}EM_n^{2+\eta}\leq d\Big[&(25a)^{2+\eta}+6(1-b)^{-1}25^{2+\eta}[n/r]EM_{2r}^{2+\eta}\\&+80(1-b)^{-1}[n\tilde{\alpha}(r)/r]^{1/q}(2+\eta)\int_{25a}^{\infty}x^{1+\eta}P^{1/p}(M_n>\tfrac{1}{25}x)\,\mathrm{d}x\Big].\end{aligned}$$

Let us analyze the integral from the right-hand side. By standard arguments involving Tchebyshev's inequality, and by the definition of p,

$$\begin{aligned}&\int_{25a}^{\infty}x^{1+\eta}P^{1/p}(M_n>\tfrac{1}{25}x)\,\mathrm{d}x\\&\quad\leq(25^{2+\delta}EM_n^{2+\delta})^{1/p}\int_{25a}^{\infty}x^{1+\eta-(2+\delta)/p}\,\mathrm{d}x\\&\quad=25^{2+\eta}a^{2+\eta-(2+\delta)/p}((2+\delta)/p-(2+\eta))^{-1}(EM_n^{2+\delta})^{1/p}.\end{aligned}$$

This completes the proof of Lemma 2.5. □

Lemma 2.6. *We assume that* $\{X_k\}$ *is a strictly stationary centered sequence of random variables such that for some* $\delta>0$, $E|X_1|^{2+\delta}<\infty$, *and* $\sigma_n^2=nh(n)$, *where* $h(x)$ *is a function which is slowly varying when* $x\to\infty$. *Assume in addition that* $\{X_k\}$ *satisfies* (1.1) *with* $\tilde{\lambda}(n)\to 0$ *as* $n\to\infty$ *and for some* $\beta>(2+\delta)/\delta$, $\tilde{\alpha}(n)=O(n^{-\beta})$ *as* $n\to\infty$. *Then there is* η, $0<\eta<\delta$ *and a constant* $K>0$ *such that*

$$E(M_n/\sigma_n)^{2+\eta}\leqslant K \quad \textit{for every } n\geqslant 1.$$

Proof. The proof follows by induction on n. First some remarks:

(i) Let us mention first that we can find a sequence $r_n=o(n)$ larger than $n/\log n$ and such that $(n/r_n)\tilde{\rho}(r_n)\to 0$ as $n\to\infty$. To see this we take first $k_n=o(n)$. Then we can find a sequence $j_n\to\infty$, $j_n=o(n)$ such that $j_n\tilde{\rho}(k_n)\to 0$ as $n\to\infty$. We now take $v_n=[n/j_n]$ and $r_n=\max(v_n, k_n, n/\log n)$.

(ii) At this point we select $\eta=\eta(\delta,\beta)$, $p=p(\delta,\beta)$ and $q=q(\delta,\beta)$ such that $\eta<\delta$, p and q satisfy the requirement from Lemma 2.5, and for $r(n)$ selected at the point (i) we have

$$\lim_{n\to\infty}[\tilde{\alpha}(r)n/r]^{1/q}[E(M_n/\sigma_n)^{2+\delta}]^{1/p}=0. \tag{2.7}$$

First we select $\eta=\eta(\delta,\beta)$ such that $(2+\eta)/(\delta-\eta)<2\beta/(2+\delta)$. Now we select a q in the interval $((2+\delta)/(\delta-\eta),\ 1+2\beta/(2+\delta))$. Note that we can easily verify now that p defined by $1/p+1/q=1$, satisfies the restriction from Lemma 2.5. Moreover, q/p belongs to the interval $((2+\eta)/(\delta-\eta),\ 2\beta/(2+\delta))$. Notice now that, by the construction of $r(n)$ and the size of $\tilde{\alpha}(n)$ we can find a constant $c\geqslant 0$ such that

$$[n\tilde{\alpha}(r)/r]\leqslant cn^{-\beta}(\log n)^{1+\beta} \quad \text{for every } n\geqslant 2.$$

Because of $\sigma_n^2=nh(n)$, with $h(n)$ a function slowly varying at ∞ we have

$$E(M_n/\sigma_n)^{2+\delta}\leqslant n^{1+\delta/2}h(n)^{-1-\delta/2}E|X_1|^{2+\delta}.$$

The last two relations imply

$$[n\tilde{\alpha}(r)/r]^{1/q}(E(M_n/\sigma_n)^{2+\delta})^{1/p}\leqslant c_1 n^{-\beta/q+(2+\delta)/2p}g(n),$$

where $c_1=c^{1/q}[E|X_1|^{2+\delta}]^{1/p}$ and $g(x)=(\log x)^{(1+\beta)/q}h(x)^{(1-\delta/2)/p}$ is a function slowly varying at ∞. In order to establish (2.7) we have only to notice that the power of n is strictly negative because q/p is strictly smaller than $2\beta/(2+\delta)$.

(iii) We shall use once again the fact that $\sigma_n^2=nh(n)$ where $h(x)$ is a slowly varying function at ∞. Using Karamata representation for slowly varying functions (see Ibragimov and Linnik, 1971, Appendix 1, 394) it is easy to see that for $\eta>0$, $0<\varepsilon<\frac{1}{2}\eta$ and $r=r(n)=o(n)$ as $n\to\infty$ we have

$$(n/2r)^{1+\varepsilon}=o(\sigma_n/\sigma_{2r})^{2+\eta} \quad \text{as } n\to\infty. \tag{2.8}$$

Now let ζ be a real number, $0<\zeta<1/(1+2\cdot 25^{2+\eta})$. Denote $b^*:=4/A^2+\zeta$ and $(d^*)^{-1}:=1-2b^*(1-b^*)^{-1}25^{2+\eta}$. Choose A sufficiently large such that

$$b^*<1 \quad \text{and} \quad d^{*\,-1}>0. \tag{2.9}$$

Denote by c_1^*, c_2^* and c_3^* the constants from Lemma 2.4 where b and d are replaced by b^* and d^*.

By remarks (i), (ii) and (iii) it is possible to choose an integer N such that for every $n > N$ we can find $r = r(n)$ satisfying:

$$[2n/r]^{1/2}\tilde{\rho}(r) < \zeta, \tag{2.10a}$$

$$\max_{2r \le i \le n} P(|S_i| > A\sigma_n) \le 2/A^2, \tag{2.10b}$$

$$n/2r \le (2r/n)^{\varepsilon}(\sigma_n/\sigma_{2r})^{2+\eta}, \tag{2.10c}$$

$$c_2^*(2r/n)^{\varepsilon} < 2^{-1}, \tag{2.10d}$$

$$c_3^* A^{2+\eta-(2+\delta)/p}[n\tilde{\alpha}(r)/r]^{1/q}(E(M_n/\sigma_n)^{2+\delta})^{1/p} < 1. \tag{2.10e}$$

Now we choose a constant K such that

$$E(M_k/\sigma_k)^{2+\eta} \le K \quad \text{for every } k \le N \tag{2.11}$$

and

$$(c_1^* A^{2+\eta} + 1)K^{-1} + 2^{-1} < 1. \tag{2.12}$$

We shall prove that (2.11) can be extended for every $n > N$ with the same constant K.

Let us assume as an induction hypothesis that (2.11) holds for all k, $N \le k \le n-1$, and let us prove (2.11) for $k = n$. We apply Lemma 2.5 to the sequence $\{X_i/\sigma_n\}_i$. By using (2.10a, b) and (2.9), because $n > N$, we have

$$\begin{aligned} E(M_n/\sigma_n)^{2+\eta} \le\ & c_1^* A^{2+\eta} + c_2^*[n/2r]E(M_{2r}/\sigma_n)^{2+\eta} \\ & + c_3^* A^{2+\eta-(2+\delta)/p}[n\tilde{\alpha}(r)/r]^{1/q}(E(M_n/\sigma_n)^{2+\delta})^{1/p}, \end{aligned}$$

whence by (2.10c, d, e), we get

$$E(M_n/\sigma_n)^{2+\eta} \le c_1^* A^{2+\eta} + 2^{-1}E(M_{2r}/\sigma_{2r})^{2+\eta} + 1.$$

Now by the induction hypothesis and (2.12) we have the desired result. □

3. An example

In this section we shall study the output of the Tukey's (1977) '3R running median' smoothing algorithm applied to a strictly stationary ρ-mixing sequence of random variables. We notice that a class of examples of ρ-mixing sequences is provided by stationary Markov processes which satisfy the L^2-norm condition (see Rosenblatt, 1971, vii, 4, p. 207).

Let $X = \{X_i\}_{i \in \mathbb{Z}}$ be a ρ-mixing sequence of random variables which is strictly stationary with the ρ-mixing coefficients $\{\rho_n\}_{n \ge 1}$, and define recurrently for $n = 0, 1, \ldots$ the sequences $X^n = \{X_k^n\}_{k \in \mathbb{Z}}$ by $X_k^0 := X_k$, $X_k^n := \operatorname{med}(X_{k-1}^{n-1}, X_k^{n-1}, X_{k+1}^{n-1})$.

By the discussions in Mallows (1979), Tukey (1977) and Theorems A and B in Bradley (1984) for our situation the limit $X_k^{\infty}(\omega)=\lim_{n\to\infty} X_k^n(\omega)$ exists almost surely and the sequence X^{∞} is strictly stationary. Let $\mathscr{P}_m(X^{\infty})$ denotes the past of the process until the moment of time m, and $\mathscr{F}_n(X^{\infty})$ the future after the moment of time n. We establish the following result:

Proposition 3.1. *Let X be a strictly stationary ρ-mixing sequence of random variables. Then, for every $A\in\mathscr{P}_0(X^{\infty})$ and $B\in\mathscr{F}_n(X^{\infty})$ we have*

$$|P(A\cap B)-P(A)P(B)|\leq \tilde{\lambda}_n(X^{\infty})(P(A)P(B))^{1/2}+\tilde{\alpha}_n(X^{\infty}),$$

where $\tilde{\lambda}_n(X^{\infty})\to 0$ as $n\to\infty$ and for every $\varepsilon>0$, $\tilde{\alpha}_n(X^{\infty})=O(n^{-2+\varepsilon})$ as $n\to\infty$.

In other words the output of the Tukey 3R smoother satisfies a decomposed mixing condition with a strong mixing part converging to 0 at a polynomial rate.

The proof of this proposition uses the following construction of 'good sets' D^* and D^{**} due to Bradley (1984).

An ordered triplet (a, b, c) of numbers will be called monotonic if either $a\leq b\leq c$ or $a\geq b\geq c$. The idea of this construction is the following: if we apply the '3R' smoother to a sequence of numbers which contains a monotonic triplet (say: $a_{n-1}\leq a_n\leq a_{n+1}$) the following iterations obtained including the output will contain at the same location a monotonic triplet too, with the same number, a_n in the middle. Moreover a_n will serve as a barrier which cannot be passed from the right or from the left. Two such barriers will assure in our situation the asymptotical independence of the past and future of the output.

Let $N\geq 3$ be an odd integer. For each integer k define the event $D_k=\{(X_{k-1}, X_k, X_{k+1})$ is monotonic$\}$.

Let $D^*=D_1\cup D_2\cup\cdots\cup D_{2NJ-N}$ and $D^{**}=D_{2NJ+1}\cup D_{2NJ+2}\cdots\cup D_{4NJ-N}$, where J is an integer.

For each integer k define the events

$$G_k^1=\{X_k<X_{k+1}\}\cap\{X_{k+N}>X_{k+N+1}\},$$

$$G_k^2=\{X_k>X_{k+1}\}\cap\{X_{k+N}<X_{k+N+1}\}.$$

For any integers $J\geq L$ which are multiples of $2N$ define the events

$$H_{J,L}^1:=G_J^1\cap G_{J+2N}^1\cap G_{J+4N}^1\cap\cdots\cap G_L^1,$$

$$H_{J,L}^2:=G_J^2\cap G_{J+2N}^2\cap G_{J+4N}^2\cap\cdots\cap G_L^2,$$

$$H_{J,L}:=H_{J,L}^1\cup H_{J,L}^2.$$

From the proof of Lemma 3.1 in Bradley (1984) it follows that $\bar{D}^*\subset H_{0,2N(J-1)}$ and $\bar{D}^{**}\subset H_{2NJ,2N(2J-1)}$. Also, by Lemma 2.4 in Bradley (1984) if $A\in\mathscr{P}_0(X^{\infty})$ and $B\in\mathscr{F}_{4JN}(X^{\infty})$ then $A_1=A\cap D^*\in\mathscr{P}_{2NJ-N+1}(X)$ and $B_1=B\cap D^{**}\in\mathscr{F}_{2NJ}(X)$.

In order to estimate the probabilities of $\bar{D}^*$ and $\bar{D}^{**}$ we chose first a real number b, $1<b<3$, and N sufficiently large odd integer such that ρ_{N-1} is small enough to satisfy the conditions

$$b(\tfrac{1}{4}+\tfrac{1}{2}\rho_{N-1})<1,$$

$$b^{-2}+\rho_{N-1}/b(\tfrac{1}{4}+\tfrac{1}{2}\rho_{N-1})\leqslant 1,$$

$$b(\tfrac{1}{4}+\tfrac{1}{2}\rho_{N-1})+\rho_{N-1}(b(\tfrac{1}{4})+\tfrac{1}{2}\rho_{N-1})^{-1}\leqslant 1.$$

Then we have:

Lemma 3.1. *For every integer* $k\geqslant 0$,

$$P(H_{0,2N(2^k-1)})\leqslant 2[b(\tfrac{1}{4}+\tfrac{1}{2}\rho_{N-1})]^{k+1}.$$

Proof. We have

$$P(H^1_{0,2N(2^k-1)})=P(G^1_0\cap G^1_{2N}\cap\cdots\cap G^1_{2N(2^k-1)}).$$

By the definition of ρ_N,

$$\begin{aligned}P(H^1_{0,0})=P(G^1_0)&=P(X_0<X_1)P(X_0>X_1)\\&\quad+\rho_{N-1}[P(X_0<X_1)P(X_0>X_1)]^{1/2}\\&\leqslant\tfrac{1}{4}+\tfrac{1}{2}\rho_{N-1}.\end{aligned}$$

For $k=1$, by the way b and N were selected, and by similar arguments as above we get

$$\begin{aligned}P(H^1_{0,2N})&\leqslant(\tfrac{1}{4}+\tfrac{1}{2}\rho_{N-1})(\tfrac{1}{4}+\tfrac{1}{2}\rho_{N-1}+\rho_{N-1})\\&\leqslant(b(\tfrac{1}{4}+\tfrac{1}{2}\rho_{N-1}))^2.\end{aligned}$$

We shall use now an induction argument. Assume

$$P(H^1_{0,2N(2^j-1)})\leqslant(b(\tfrac{1}{4}+\tfrac{1}{2}\rho_{N-1}))^{j+1}\quad\text{for every integer } j<k.$$

We have

$$P(H^1_{0,2N(2^k-1)})\leqslant P(H^1_{0,2N(2^{k-1}-1)})(P(H^1_{0,2N(2^{k-1}-1)})+\rho_{N-1}).$$

Whence, by the induction hypothesis, and our choice of b and N, for every $k\geqslant 2$, we have

$$\begin{aligned}P(H^1_{0,2N(2^k-1)})&\leqslant(b(\tfrac{1}{4}+\tfrac{1}{2}\rho_{N-1}))^k((b(\tfrac{1}{4}+\tfrac{1}{2}\rho_{N-1}))^k+\rho_{N-1})\\&\leqslant(b(\tfrac{1}{4}+\tfrac{1}{2}\rho_{N-1}))^{k+1}.\qquad\square\end{aligned}$$

Proof of Proposition 3.1. Let $A \in \mathscr{P}_0(X^\infty)$ and $B \in \mathscr{F}_{2N2^j}(X^\infty)$ where N is as large as required by Lemma 3.1 and j is an integer. Then, according to the construction of D^* and D^{**}, it follows that $A_1 = A \cap D^* \in \mathscr{P}_{2N2^j-N+1}(X)$ and $B_1 = B \cap D^{**} \in \mathscr{F}_{2N2^j}(X)$. If is easy to see that

$$|P(A \cap B) - P(A)P(B)| \leq |P(A_1 \cap B_1) - P(A_1)P(B_1)| + 2P(\bar{D}^*).$$

Now by the properties of X and Lemma 3.1 it follows

$$|P(A \cap B) - P(A)P(B)| \leq \rho_{N-1}(X)[P(A)P(B)]^{1/2} + 4[b(\tfrac{1}{4}+\tfrac{1}{2}\rho_{N-1})]^{j+1}. \quad (3.1)$$

Now if $\log n$ denotes logarithm with the base 2, we choose N = the largest odd integer $\leq \log n$ and $j = [\log n - \log(\log n)] - 1$. By (3.1) if follows:

$$|P(A \cap B) - P(A)P(B)| \leq \tilde{\lambda}_n(X^\infty)(P(A)P(B))^{1/2} + \tilde{\alpha}_n(X^\infty)$$

for every $A \in \mathscr{P}_0(X^\infty)$, $B \in \mathscr{F}_0(X^\infty)$, where $\tilde{\lambda}_n(X^\infty) = \rho_{[\log n]-1}(X)$ and $\tilde{\alpha}(X^\infty) = 4(b(\frac{1}{4}+\frac{1}{2}\rho_{[\log n]-1}))^{[\log n - \log\log n]}$. The result of Proposition 3.1 follows because for n sufficiently large b can be selected as close to 1 as we need to assure $\tilde{\alpha}_n(X^\infty) = \mathrm{O}(n^{-\beta})$ as $n \to \infty$ for β a fixed positive real, $\beta < 2$. □

As a corollary to Theorem 1.1 we obtain the following C.L.T. for the Tukey '3R' smoother.

Proposition 3.2. *Assume that X is a strictly stationary, ρ-mixing sequence of random variables which is centered and for certain $r > 4$, $E|X_1|^r < \infty$. Let X^∞ be the output of the Tukey 3R smoother and denote by $S_n^\infty = X_1^\infty + X_2^\infty + \cdots + X_n^\infty$. Assume that $(\sigma_n^\infty)^2 = \mathrm{Var}(S_n^\infty) = nh(n)$ where $h(x)$ is a function slowly varying at infinite. Then*

$$\frac{S_{[nt]}^\infty - [nt]EX_1^\infty}{\sigma_n^\infty} \Rightarrow W.$$

Proof. It is easy to see that for each $t > 0$, $P(|X_1^\infty| < t) \leq 3P(|X_1| > t)$. Therefore $E|X_1^\infty|^r < \infty$. Now let $0 < \varepsilon < (r-4)/(r-2)$. By Proposition (3.1) $\tilde{\alpha}_n(X^\infty) \leq \mathrm{O}(n^{-2+\varepsilon})$ and $\tilde{\lambda}_n(X^\infty) \to 0$. The result of this Proposition follows from Theorem 1.1. □

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